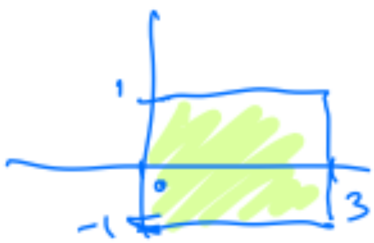
Multivariable Numerical Integration.

Method 1 : Do everything as iterated integrals, do one of the 1-variable methods.

Simple Example: $\displaystyle\int_{-1}^{1}\int_{0}^{3} x^2y^2\,dx\,dy$



$$= \int_{-1}^{1}\left(\int_0^3 x^2y^2\,dx\right)dy$$

I will use Left(2) to compute.

Inside integral $\displaystyle\int_0^3 x^2y^2\,dx$ (y constant.)

x is variable

$$\text{Left}(2) = h(y_0+y_1)$$

$$= \frac{3}{2}\left(x^2y^2\Big|_{x=0} + x^2y^2\Big|_{x=3/2}\right)$$

$$= \frac{3}{2}\left(0 + \frac{9}{4}y^2\right) = \frac{27}{8}y^2$$

$\Rightarrow$ whole integral

$$= \int_{-1}^{1}\frac{27}{8}y^2\,dy$$

$$\text{Left}(2) = \underbrace{1}_{\Delta y}\left(\frac{27}{8}y^2\Big|_{y=-1} + \frac{27}{8}y^2\Big|_{y=0}\right)$$

$$= 1\left(\frac{27}{8}(1) + 0\right) = \boxed{\frac{27}{8}}$$

Left(2) approx to integral.

Exact: $\int_{-1}^{1}\int_{0}^{3} x^2y^2\,dx\,dy$

$$= \int_{-1}^{1} \frac{x^3}{3}y^2\Big|_{x=0}^{3}\,dy = \int_{-1}^{1} 9y^2\,dy = 3y^3\Big|_{-1}^{1}$$

$$= 3-(-3) = \boxed{6}.$$

$$\text{Error}(2) = 6 - \frac{27}{8} = \frac{48-27}{8} = \frac{21}{8}.$$

We could use any of the integration methods & do the same.

Issue: Amount time taken for 1 dimension $= T$
for N dimensions $\to$ much longer.

If $N =$ (# of operations needed for 1 dim)
then $N^d =$ # of operations needed for d dimensions.

Method 2: Think of $\iint_{\text{set}} f(x,y,\dots)$

$= (\text{Average value of } f(x,y,\dots))\cdot(\text{size of the set})$.

e.g. $\int_a^b f(x)\,dx = \left(\text{avg value of } f(x) \text{ on } [a,b]\right)\cdot(b-a)$

$$\iint_S g(x,y)\,dx\,dy = \begin{pmatrix}\text{avg value of}\\ g(x,y) \text{ on } S\end{pmatrix}\cdot\begin{pmatrix}\text{Area}\\ \text{of } S\end{pmatrix}$$

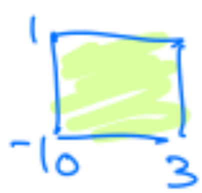
Monte Carlo Integration Method

$$\iint_S \cdots\, g(x,y,\cdots)\,dx\,dy\cdots \approx \begin{pmatrix}\text{Avg of } g(x,y)\\ \text{over a large \#}\\ \text{of random pts}\\ \text{in } S\end{pmatrix}\cdot\begin{pmatrix}\text{Size}\\ \text{of}\\ S\end{pmatrix}.$$

This works in <u>any</u> dimension, even 1 dimension.

But, it's slow.

<u>Example:</u> $\int_{-1}^{1}\int_{0}^{3} x^2y^2\,dx\,dy = I$

$S =$ 

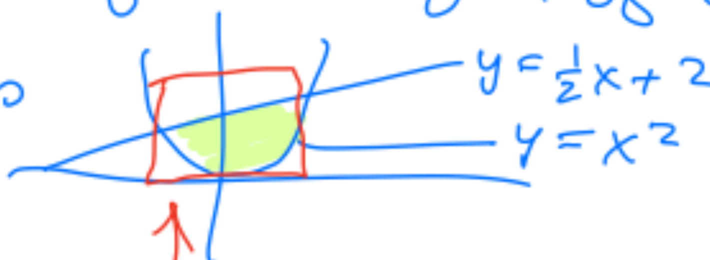
$\Rightarrow \text{Area}(S) = 6$

$$I \approx \left(\begin{array}{c} \text{avg of } x^2y^2 \text{ over} \\ \text{randomly chosen} \\ \text{pts in } [0,3]\times[-1,1] \end{array} \right) \cdot 6$$

Let's use sagemath to compute the random values.

with 10,000 pts → gives a pretty good answer.

What if the region of integration was



?

Solution: put a box around region.

Only compute the function & count that point if

$$x^2 \le y \le \frac{1}{2}x + 2.$$
